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Additive bijections of $C(X)$

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Abstract

For compact Hausdorff spaces X and Y, the Stone–Banach theorem asserts that surjective linear isometries $H:C(X)\to C(Y)$ are of the form $Hf(y)=f(h(y))H\mathbf{1}(y)$ ($f\in C(X)$, $y\in Y$ and $\mathbf{1}(x)\equiv 1$) where $h:Y\to X$ is a homeomorphism and $|H\mathbf{1}(y)|\equiv 1$. Omitting the requirements that h be a homeomorphism and that $|H\mathbf{1}(y)|\equiv 1$, maps of this type $f\mapsto (f\circ h)H\mathbf{1}$ are called 'weighted composition maps' where $H\mathbf{1}\in C(Y)$ is the 'weight' function. Instead of $K=\mathbb{R}$ or $\mathbb{C}$, suppose $(K,|\,|)$ is a valued field. We now consider K-valued continuous functions $C(X,K)$ and $C(Y,K)$. Now linear isometries $H:C(X,K)\to C(Y,K)$ may take different forms. Indeed, if $(K,|\,|)$ is non-Archimedean (i.e., $|a+b|\leqslant \max(|a|,|b|)$), a linear isometry $H:C(X,K)\to C(Y,K)$ is a weighted composition if and only if it is *separating* in the sense that, for all $f,g\in C(X,K)$, $fg=0\Rightarrow HfHg=0$. We weaken linear to additive and isometry to separating bijection and consider what forms such a bijection $H:C(X,K)\to C(Y,K)$ can have for $K=\mathbb{R}$, $\mathbb{C}$ or a non-Archimedean valued field. We show in Theorem 18 that an additive separating bijection $H:C(X,K)\to C(Y,K)$ is automatically continuous; it is a weighted composition map with a homeomorphism if $K=\mathbb{R}$ or $\mathbb{Q}_p$ the p-adic numbers) and 'almost' a weighted composition if $K=\mathbb{C}$ (see Theorem 18(b)). © 1999 Elsevier Science B.V. All rights reserved.

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1. Valued fields

As our standard reference on rings of continuous functions, we take [10]; for valuation theory, see [4]; for functional analysis over valued fields, see [15,17,13] or the survey article [14]. A *valued field* $(K,|\,|)$ is a field K with a *valuation* $|\,|:K\to\mathbb{R}$, where $|\,|$ is such

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that, for all $a,b\in K$, $|a|\geqslant 0$, $=0$ if and only if $a=0$, $|ab|=|a|\,|b|$ and $|a+b|\leqslant|a|+|b|$. If $|\,|$ satisfies the *ultrametric inequality*, $|a+b|\leqslant\max(|a|,|b|)$ for all $a,b\in K$, it is called *non-Archimedean*; otherwise it is called *Archimedean*. If $|\,|$ is non-Archimedean then the open and closed balls $B(0,r)=\{a\in K\colon |a|<r\}$ and $C(0,r)=\{a\in K\colon |a|<r\}$, $r>0$, respectively, about 0 are clopen; consequently non-Archimedean valuations induce 0-dimensional topologies.

As examples of non-Archimedean valuations, consider the *trivial valuation*: $|a|=1$ for all $a\neq 0$, $|0|=0$ on any field K.

Example 1. Choose a prime p, write any rational number $x=p^q(c/d)$ where c,d and q are integers and p is not a factor of c or d and take $|x|=p^{-q}$. This defines the *p-adic valuation* $|\,|_p$ on the rationals $\mathbb{Q}$. The metric completion $\mathbb{Q}_p$ of $(\mathbb{Q},|\,|_p)$ is called the field of *p-adic numbers*. As $\mathbb{Q}_p$ and $\mathbb{R}$ are each completions of $\mathbb{Q}$, albeit with respect to different metrics, they have certain similarities. For example, any element x of $\mathbb{Q}_p$ or $\mathbb{R}$ can be expressed as an infinite 'decimal': for $x\in\mathbb{Q}_p$, $x=\sum_{j\geqslant n}a_jp^j$ where n are a_j are integers [4, p. 35]. Differences can be profound, however. In $\mathbb{Q}_5$, for example, there is a solution to $x^2+1=0$, i.e., you could say that $\mathrm{i}=\sqrt{-1}\in\mathbb{Q}_5$. There is also significant variability between the $\mathbb{Q}_p$ for different primes. Indeed, for distinct primes p and q, $\mathbb{Q}_p$ is not field-isomorphic to $\mathbb{Q}_q$ [4, p. 61].

Example 2. Let F be any field, let x be an indeterminate over F and consider the set $F(x)$ of formal Laurent series $\sum_{n\geqslant k}a_nx^n$ where the $a_n\in F$ and k is an integer (possibly negative). $F(x)$ is a field with respect to pointwise addition and Cauchy product. Choose $r\in(0,1)$ and define $|\sum_{n\geqslant k}a_nx^n|=r^k$. $(F(x),|\,|)$ is a non-Archimedean valued field.

Definition 3. By an *integer* n in the field K, we mean $\pm(1+1+\cdots+1)$ (n summands). For integers m and $n\neq 0$, the elements mn^{-1} are called the *rational* elements of $\dot{K}$. We denote the set of integers and rationals of K by $\mathbb{Z}$ and $\mathbb{Q}$, respectively.

If K is finite, so is $\mathbb{Q}$. In some important cases ($\mathbb{Q}_p$, for example) $\mathbb{Q}$ is dense in K.

The following result is part of the folklore of valuation theory.

Theorem 4. *If $(K,|\,|)$ is non-Archimedean and the restriction of $|\,|$ to the integers of K is nontrivial, then $|\,|$ is nontrivial on the rationals $\mathbb{Q}$ of K as well. When this happens, $|\,|$ must be a p-adic valuation on $\mathbb{Q}$ for some prime p.*

Proof. By the ultrametric inequality (i.e., $|a+b|\leqslant\max(|a|,|b|)$), for any $n\in\mathbb{N}$, $|n|\leqslant 1$. Let p be the least positive integer such that $|p|<1$. We show that p is prime. If $p=rs$, $1<r<p$, then $|p|=|r||s|<1$; hence $|r|<1$ or $|s|<1$. This contradicts the fact that p is the least integer whose absolute value is less than 1. Next, we show that if p and the positive integer n are relatively prime then $|n|=1$. Since p does not divide n there exists positive integer k and r such that $n=kp+r$ and $1\leqslant r<p$. Since $r<p$, $|r|=1$; in addition, $|kp|=|k|\,|p|<1$. Since $|\,|$ is non-Archimedean and $|r|=1$ it follows from the ultrametric inequality that $|kp + r| = \max(|kp|, |r|) = 1$. Now let $m = p^q b,\ q \geqslant 0$, where p is not a divisor of b, be any positive integer. Since $|b| = 1$, then $|m| = |p|^q |b| = |p|^q$. It follows that $|\,| = |\,|_p$. $\square$

2. Setting/notation

Let $C(X, K)$ and $C(Y, K)$ denote the spaces of K-valued continuous functions on the Tychonoff spaces X and Y, where $(K, |\,|)$ is a valued field.

Definition 5. Let D be a subalgebra of $C(X, K)$. A map $H : D \to C(Y, K)$ is called *separating* if, for all $f, g \in D$, $fg = 0 \Rightarrow Hf\,Hg = 0$. H is *biseparating* if it is bijective and H and H^{-1} are separating ($fg = 0$ if and only if $Hf\,Hg = 0$).

Examples of separating maps include differentiation, ring isomorphism, and weighted composition. Integration is not separating since, for $K = \mathbb{R}$, it maps disjoint hat functions into eventually constant functions, viz., let $f(x) = (1 - |x|)k_{[-1,1]}$ and $g(x) = (1 - |x - 2|)k_{[1,3]}$ where $k_{[-1,1]}$ and $k_{[1,3]}$ denote the characteristic functions of $[-1, 1]$ and $[1, 3]$, respectively, and let $Hf(x) = \int_{-1}^{x} f(t)\,\mathrm{d}t$; then $fg = 0$ but $Hf\,Hg \neq 0$. Some basic properties of separating maps are listed in Theorem 8.

Definition 6. Let $w \in C(Y, K)$ and call w a *weight* function. Let $g : Y \to X$ be continuous. A map $H : C(X, K) \to C(Y, K)$, of the form $f \mapsto (f \circ g)w$ is called a *weighted composition map* with *weight function* w.

In several previous articles ([2] and [6], for example) various properties of *linear* separating maps have been established. For example, if X and Y are Tychonoff spaces and H is a linear biseparating map then the realcompactification υX is homeomorphic to υY; if X and Y are realcompact then H is the weighted composition $Hf = (f \circ h) \cdot H\mathbf{1}$ [1, Propositions 2 and 3], and H is automatically continuous when $C(X, K)$ and $C(Y, K)$ carry the compact-open topology. In this paper we investigate the form and automatic continuity of additive (as opposed to linear) separating maps. We pay special attention to automatic continuity results for linear separating H that survive for additive H. The main result is Theorem 8 on additive separating bijections. Such a map must be automatically continuous, biseparating and almost a weighted composition with a homeomorphism h; if $K = \mathbb{R}$ or $\mathbb{Q}_p$ then H must be a weighted composition $Hf = (f \circ h) \cdot H\mathbf{1}$. If $K = \mathbb{C}$ and we sharpen 'bijection' to isometry with Y connected, we show in Theorem 19 that H is either the weighted composition $Hf = (f \circ h) \cdot H\mathbf{1}$ or $Hf = \overline{(f \circ h)} \cdot H\mathbf{1}$ where $h : Y \to X$ is a homeomorphism. We demonstrate the necessity of certain hypotheses by means of examples in Section 5.

We adhere to the following notations and standing assumptions throughout.

(1) X and Y denote compact Hausdorff spaces.

(2) $K = \mathbb{R}$, $\mathbb{C}$ or a nontrivially valued non-Archimedean field. When K is non-Archimedean and $K \neq \mathbb{Q}_p$, we assume that the restriction of $|\,|$ to $\mathbb{Z}$ is nontrivial.

(3) $C(X,K)$ and $C(Y,K)$ denote the sup-normed spaces of continuous K-valued functions on X and Y, respectively. If K is non-Archimedean, we assume that X and Y are 0-dimensional in the sense that each topology has a base of clopen sets.
(4) $\mathbf{1}$ stands for the function in $C(X,K)$ which is 1 at each $x \in X$; $[\mathbf{1}] = \{a\mathbf{1}\colon a \in K\}$ denotes the linear span of $\mathbf{1}$.
(5) For any function f, $\operatorname{coz} f$ denotes the cozero set of f; the topological closure of a set U is denoted $\operatorname{cl} U$.
(6) $H : C(X,K) \to C(Y,K)$ denotes an additive separating map such that, for each $y \in Y$, there exists $f \in C(X,K)$ such that $Hf(y) \neq 0$.
(7) For $y \in Y$, the function $y\wedge \circ H$ is defined at each $f \in C(X,K)$ and $y \in Y$ by $y\wedge \circ H(f) = Hf(y)$, i.e., $y\wedge \circ H(f)$ is the evaluation of Hf at y.
(8) The *support* of $y\wedge \circ H$ is the singleton (Theorem 8(a))

$$\operatorname{supp} y\wedge \circ H = \bigcap_{Hf(y)\neq 0} \operatorname{cl}\operatorname{coz} f.$$

(9) The *support map* of an additive separating map H is the continuous map $h : Y \to X$ defined by $h(y) = \operatorname{supp} y\wedge \circ H$; we reserve the letter h for the support map of H everywhere in the sequel.

3. Separating maps—basics

We list some basic properties of the separating map H and its support map h in Theorem 8. Proofs for linear H can be found in [3,8] and [11] for $K = \mathbb{R}$ or $\mathbb{C}$ and [2, Theorems 3.2, 3.3, and 4.1] in the case when K is non-Archimedean. The results remain true for additive H and are easily adapted to that case.

Definition 7. We say that the separating map H is *detaching* if for any two distinct points $y, y' \in Y$ there exist $f, g \in C(X,K)$ such that $\operatorname{cl}\operatorname{coz} f$ and $\operatorname{cl}\operatorname{coz} g$ are disjoint while $Hf(y)Hg(y') \neq 0$.

The notion of detaching is important because it makes the support map h injective (Theorem 8(e)).

Theorem 8. *Let $H : C(X,K) \to C(Y,K)$ denote an additive separating map as in Section 2. Let $f, g \in C(X,K)$.*
(a) *Let $y\wedge \circ H$ be as in (7) of Section 2. Then, for each $y \in Y$, $\operatorname{supp} y\wedge \circ H = \bigcap_{Hf(y)\neq 0} \operatorname{cl}\operatorname{coz} f$ is a singleton. Also, the* support map *h of H (Sections 2, 9) is continuous.*
(b) *If $f = 0$ on an open subset U of X, then $Hf = 0$ on $h^{-1}(U)$; equivalently, if $f = g$ on U, then $Hf = Hg$ on $h^{-1}(U)$.*
(c) *For any f, $h(\operatorname{coz} Hf) \subset \operatorname{cl}\operatorname{coz} f$.*
(d) *If H is injective, then $h(Y)$ is dense in X.*
(e) *The support map h is injective if and only if H is detaching.*

As $C(X,K)$ is a metric space, we may consider continuity of the maps $y^\wedge \circ H$ $(y \in Y)$.

Definition 9. Let $H : C(X,K) \to C(Y,K)$ denote an additive separating map. We call the set $Y_c = \{y \in Y\colon\ y^\wedge \circ H \text{ is continuous}\}$ the set of *continuity points* of H. The complement Y_d of Y_c is called the set of *discontinuity points* of H. If $Y = Y_c$ we say that H is *pointwise continuous*.

For $y \in Y_c$, $Hf(y)$ has a simple form; as we show in Theorem 10, it is almost a weighted composition. We sharpen things somewhat in Proposition 11 (it *is* a weighted composition for $K = \mathbb{R}$ or $\mathbb{Q}_p$). We show that Y_c is closed in Proposition 14.

Theorem 10. *$H : C(X,K) \to C(Y,K)$ denotes an additive separating map and let h be the support map of H. If $y \in Y_c$, then $Hf(y) = H[f(h(y))\mathbf{1}](y)$ for all $f \in C(X,K)$.*

Remark. $y^\wedge \circ H$ need not be continuous for $Hf(y)$ to have this form. In Example 23 we exhibit a discontinuous additive (but not linear) separating map H such that $[y^\wedge \circ H](f) = Hf(y) = H[f(h(y))\mathbf{1}](y)$ for all $f \in C(X,K)$.

Proof. (Cf. [8, Theorem 2.2].) We show first that if $f(h(y)) = 0$ $(f \in C(X,K),\ y \in Y)$ then $Hf(y) = 0$. For each $n \in \mathbb{N}$, let U_n be an open neighborhood of $h(y)$ such that $\sup|f(\operatorname{cl} U_n)| < 1/n$ and let V_n be an open neighborhood of $h(y)$ such that $\operatorname{cl} V_n \subset U_n$. Now choose $g_n \in C(X,K)$ $(n \in \mathbb{N})$ such that $0 \leqslant |g_n| \leqslant 1$, $g_n(\operatorname{cl} V_n) = 1$ and $g_n(X \setminus U_n) = 0$. Clearly $fg_n \to 0$. Since $f = fg_n$ on V_n then $Hf = Hfg_n$ on $h^{-1}(V_n)$ by Theorem 8(b). In particular, $Hf(y) = H(fg_n)(y)$. Since $fg_n \to 0$ and $y^\wedge \circ H$ is continuous it follows that $H(fg_n)(y) \to 0$. Therefore $Hf(y) = 0$. Now suppose that $f(h(y)) \neq 0$ and consider $g = f - f(h(y))\mathbf{1}$. Since $g(h(y)) = 0$, it follows that $0 = Hg(y) = Hf(y) - H[f(h(y))\mathbf{1}](y)$. $\square$

We elaborate further on the theme of Theorem 10, namely on how continuity of $y^\wedge \circ H$ determines the form of $Hf(y)$, in Proposition 11.

Proposition 11. *Let $H : C(X,K) \to C(Y,K)$ be an additive separating map and let h be the support map of H. For any $f \in C(X,K)$ and $y \in Y_c$,*

(a) *if K is $\mathbb{R}$ or $\mathbb{Q}_p$, then $Hf(y) = f(h(y))H\mathbf{1}(y)$;*

(b) *if $K = \mathbb{C}$, then*

$$\begin{aligned} Hf(y) &= H\big[f(h(y))\mathbf{1}\big] \\ &= \big(\operatorname{Re}\big[f(h(y))\big]\big)H\mathbf{1}(y) + \big(\operatorname{Im}\big[f(h(y))\big]\big)H(\mathrm{i}\mathbf{1})(y). \end{aligned}$$

Proof. (a) By Theorem 10, $Hf(y) = H[f(h(y))\mathbf{1}](y)$. In $\mathbb{R}$ or $\mathbb{Q}_p$ there is a sequence $r_n \in \mathbb{Q}$ such that $r_n \to f(h(y))$. Since H is additive, $H[r_n\mathbf{1}](y) = r_n H\mathbf{1}(y)$; continuity of $y^\wedge \circ H$ yields the result.

(b) For $K = \mathbb{C}$, by Theorem 10,

$$[y^\wedge \circ H](f) = Hf(y) = H\bigl[\operatorname{Re} f(h(y) + \mathrm{i} \operatorname{Im} f(h(y))\mathbf{1}\bigr](y).$$

Together with (a) and the additivity of H, this completes the proof. □

As noted in Theorem 8(e), the support map h of H is injective (on Y) if and only if H is detaching. As we show next, a weaker separation condition on $H(C(Y, K))$ suffices for h to be injective on Y_c.

Proposition 12. *$H : C(X, K) \to C(Y, K)$ denotes an additive separating map. Let $H(C(X, K))$ separate points of Y in the sense that for $y \neq y'$ in Y there exists $f \in C(X, K)$ such that $Hf(y) = 1$ while $Hf(y') = 0$. Furthermore*:

(i) *If $K = \mathbb{Q}_p$ for some prime p, or $K = \mathbb{R}$, then $H\mathbf{1}(y) \neq 0$ for every $y \in Y$.*

(ii) *If $K = \mathbb{C}$ or K is non-Archimedean and $K \neq \mathbb{Q}_p$ for any prime p, then $H(a\mathbf{1})(y) \neq 0$ for all nonzero $a \in K$ and $y \in Y$.*

Then the support map h of H is injective on Y_c.

Proof. For $K = \mathbb{R}$, or $\mathbb{Q}_p$, this follows immediately from (i) and (ii) and the forms that $Hf(y)$ $(y \in Y_c)$ has, enunciated in Proposition 11.

If K is $\mathbb{C}$ or non-Archimedean and $K \neq \mathbb{Q}_p$, suppose $y \neq y' \in Y_c$. Since $H(C(X, K))$ separates points of Y, there exists $f \in C(X, K)$ such that $Hf(y) = H[f(h(y))\mathbf{1}](y) = 1$ and $Hf(y') = H[f(h(y')\mathbf{1}](y') = 0$ by Theorem 10. Since $Ha\mathbf{1}$ only vanishes if $a = 0$ by (ii), the second equation implies that $f(h(y')) = 0$ while the first implies that $f(h(y)) \neq 0$. It follows that $h(y) \neq h(y')$. □

In order to prove Proposition 14, we need an analog of a well-known criterion for continuity of *linear* (they are only additive in Proposition 13 below) maps on normed spaces, namely

Let E and F be normed linear spaces over a nontrivially valued field K. A linear map $A : E \to F$ is continuous if and only if $\sup_{x \neq 0} \|Ax\|/\|x\| < \infty$ [15, Lemma 1, p. 77].

Proposition 13. *Let E and F be normed spaces over K. An additive map $A : E \to F$ is continuous if and only if $\sup_{x \neq 0} \|Ax\|/\|x\| < \infty$.*

Proof. If $\sup_{x \neq 0} \|Ax\|/\|x\| < \infty$, there exists $k > 0$ such that $\|Ax\| < k\|x\|$ for all $x \in E$. Thus, A is continuous at 0; since A is additive, it is continuous at all $x \in E$.

As to the converse, we consider the Archimedean and non-Archimedean cases separately; we prove the contrapositive in each case.

Let K be $\mathbb{R}$ or $\mathbb{C}$ and suppose that there exists a sequence (x_n) in E such that $\|Ax_n\|/\|x_n\| \geqslant n^2$, for each n. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, so is $\|x_n\|\mathbb{Q}$ for each n; consequently, for each n, there exists a positive $r_n \in \mathbb{Q}$ such that $2 \geqslant \|r_n x_n\| \geqslant 1$. Thus,

$\|Ar_nx_n\|/\|r_nx_n\| = \|Ax_n\|/\|x_n\| \geqslant n^2$, while $2/n \geqslant \|r_nx_n\|/n \geqslant 1/n$. We conclude from the previous inequality that $\|r_nx_n\|/n \to 0$. Since

$$\frac{\|Ar_nx_n/n\|}{\|r_nx_n/n\|} \geqslant n^2$$

it follows that $\|Ar_nx_n/n\| \geqslant n^2\|r_nx_n/n\| \geqslant n^2(1/n) = n$. Hence A is not continuous and this establishes the result for $K = \mathbb{R}$ or $\mathbb{C}$.

Now suppose that K is non-Archimedean. Since the restriction of $|\,|$ to the integers of K is nontrivial (see Section 2), $|\,|$ must be a p-adic valuation for some prime p by Theorem 4. Let $a \in K$ be such that $r = |a| > 1$ is a generator of the multiplicative group $|\mathbb{Q}_p^*| = \{|a|\colon a \in \mathbb{Q}_p - \{0\}\}$. (The generator $r = 1/p$ in this case.) Suppose $x_n \in E$ is such that $\|Ax_n\|/\|x_n\| \geqslant r^{2n+1}$. If $\|x_1\| \geqslant 1/r$, since $r^{-n} \to 0$, there exists a smallest positive integer k_1 such that $\|x_1\|/r^{k_1} \leqslant 1/r$. Since k_1 is the smallest, $\|x_1\|/r^{k_1-1} > 1/r$. Therefore $\|x_1\|/r^{k_1} > 1/r^2$ or

$$\frac{1}{r^2} < \|a^{-k_1}x_1\| \leqslant \frac{1}{r}. \tag{1}$$

If $\|x_1\| < 1/r$ and $\|x_1\| \leqslant 1/r^2$, choose the first positive integer k_1 such that $r^{k_1}\|x_1\| > 1/r^2$. Then $r^{k_1-1}\|x_1\| \leqslant 1/r^2$ and $r^{k_1}\|x_1\| < 1/r$ and inequality (1) is satisfied with k_1 replaced by $-k_1$. It is now clear that there is a sequence of integers k_n such that $r^{-(n+1)} \leqslant \|a^{k_n}x_n\| \leqslant r^{-n}$ for every n. Since A is additive and $a_n \in \mathbb{Q} \subset K$,

$$\frac{\|A(a^{k_n}x_n)\|}{\|a^{k_n}x_n\|} = \frac{\|a^{k_n}Ax_n\|}{\|a^{k_n}x_n\|} = \frac{\|Ax_n\|}{\|x_n\|} > r^{2n+1}.$$

Thus

$$\bigl\|A(a^{k_n}x_n)\bigr\| > \|a^{k_n}x_n\|r^{2n} \geqslant r^{(n-1)} \to \infty.$$

Since $\|a^{k_n}x_n\| \leqslant r^{-n} \to 0$, it follows that A is not continuous. $\square$

Proposition 14. *$H : C(X,K) \to C(Y,K)$ denotes an additive separating map. If*
(a) *$K = \mathbb{R}$, $\mathbb{Q}_p$, or $\mathbb{C}$, or*
(b) *$K \neq \mathbb{Q}_p$ is non-Archimedean and $H|_{[\mathbf{1}]}$ is continuous,*
then Y_c is closed in Y.

Proof. Let (y_a) be a net of points from Y_c such that $y_a \to y \in Y$. Let h be the support map of H.

(a) When $K = \mathbb{R}$ or $\mathbb{Q}_p$, then, for any $f \in C(X,K)$,

$$Hf(y_a) = H\bigl[f\bigl(h(y_a)\bigr)\mathbf{1}\bigr](y_a) = f\bigl(h(y_a)\bigr)H\mathbf{1}(y_a) \to f\bigl(h(y)\bigr)H\mathbf{1}(y)$$

(Proposition 11). Also, $Hf(y_a) \to Hf(y)$. Thus $Hf(y) = f(h(y))H\mathbf{1}(y)$ and it follows from this form that $y \in Y_c$. When $K = \mathbb{C}$, $Hf(y)$ is of the form of Proposition 11(b) and we can apply the previous argument to the real and imaginary parts of f.

(b) By the continuity of Hf, $Hf(y_a) \to Hf(y)$. By Theorem 10

$$Hf(y_a) = H\bigl[f(h(y_a))\mathbf{1}\bigr](y_a)$$

for each index a. Since $H|_{[\mathbf{1}]}$ is continuous, it follows that $H[f(h(y_a))\mathbf{1}] \to H[f(h(y))\mathbf{1}]$. Hence, $Hf(y) = [Hf(h(y))\mathbf{1}](y)$. By Proposition 13, there exists $C > 0$ such that

$$\left|Hf(y)\right| = \left|H\left[f\left(h(y)\right)\mathbf{1}\right](y)\right| \leqslant C\left\|f\left(h(y)\right)\mathbf{1}\right\| = C\left|f\left(h(y)\right)\right|\|\mathbf{1}\| \leqslant C\|f\|.$$

It follows that $y \in Y_c$. □

Proposition 15 was obtained for $\mathbb{R}$- and $\mathbb{C}$-valued functions in [11].

Proposition 15. *$H : C(X, K) \to C(Y, K)$ denotes an additive separating map and h denotes the support map of H. Then $h(Y_d)$ is finite.*

Proof. Assume that $h(Y_d)$ is infinite. As any infinite subset of a completely regular space contains a discrete subset, we may assume that $\{h(y_n)\}$ is discrete. Since X is compact, by [9, Theorem 2.1.14], there exists a sequence of pairwise disjoint open neighborhoods V_n of $h(y_n)$. Since X is Tychonoff, for each n, there exists an open neighborhood $U_n \subset V_n$ of $h(y_n)$ such that $\operatorname{cl} U_n \subset V_n$.

If K is $\mathbb{R}$ or $\mathbb{C}$, choose continuous $\mathbb{R}$-valued functions k_n, $\|k_n\| = 1$, such that $k_n(U_n) \equiv 1$ and $\operatorname{coz} k_n \subset V_n$. By Proposition 13 there exists $f_n \in C(X, K)$, $\|f_n\| = 1$, such that $|Hf_n(y_n)|/\|f_n\| = |Hf_n(y_n)| \geqslant n^3$. Let $g_n = f_n/n^2$. By Theorem 8(b), since $g_n|_{U_n} = g_n k_n|_{U_n}$, it follows that $|Hg_n(y_n)| = |H(g_n k_n)(y_n)|$ for every $n \in \mathbb{N}$. Hence

$$\left|Hg_n(y_n)\right| = \frac{1}{n^2}\left|Hf_n(y_n)\right| \geqslant n, \quad n \in \mathbb{N}$$

Define $g = \sum_{n\in\mathbb{N}} k_n g_n$. Since each $\|k_n g_n\| \leqslant 1/n^2$, the series converges 'absolutely' so $g \in C(Y)$.

We show next the contradictory result that Hg is unbounded on the compact set Y. Since the $k_n g_n$ have pairwise disjoint cozero sets, for any $m \in \mathbb{N}$, $k_n g_n(y_m) = 0$ for $n \neq m$; therefore $H(k_n g_n)(y_m) = 0$ for $n \neq m$, by Theorem 8(b). Thus, $Hg(y_m) = H(k_m g_m)(y_m) > m$.

If K is non-Archimedean we take X and Y to be 0-dimensional (see Sections 2, 3) so we may take pairwise disjoint *clopen* neighborhoods U_n of $h(y_n)$, $n \in \mathbb{N}$, and take k_n to be the K-valued characteristic function of U_n. Choose $f_n \in C(X, K)$, $\|f_n\| = 1$, such that

$$\frac{|Hf_n(y_n)|}{\|f_n\|} = \left|Hf_n(y_n)\right| \geqslant n^3.$$

Choose $a \in \mathbb{Q} \subset K$ such that $|a| \geqslant 2$ and let $g_n = f_n/a^n$. Now use the argument above. □

Proposition 16. *$H : C(X, K) \to C(Y, K)$ denotes an additive injective separating map and let h be the support map of H. Then if*

(a) *X has no isolated points or*

(b) *X has isolated points and $H|_{[\mathbf{1}]}$ is continuous*

then $h(Y_d) \subset h(Y_c)$.

Proof. (a) Since H is injective, $\operatorname{cl} h(Y) = X$ by Theorem 8(b). Since Y is compact, $h(Y) = X$. Thus, $X = h(Y) = h(Y_c) \cup h(Y_d)$. Since $h(Y_d)$ is finite (Proposition 15), it is closed. As Y_c is closed (Proposition 14), it follows that $h(Y_c)$ is closed. Now suppose that $y_0 \in Y_d$ is such that $h(y_0) \notin h(Y_c)$. If so, then $\{h(y_0)\}$ is clopen which cannot happen if X has no isolated points and (a) is proved.

(b) Suppose that $H|_{[\mathbf{1}]}$ is continuous. By Proposition 14, Y_c (and therefore $h(Y_c)$) is closed. Suppose $y_0 \in Y_d$ and $h(y_0) \notin h(Y_c)$. Since, for any $f \in C(X, K)$, $f = f(h(y_0))\mathbf{1}$ on the open set $\{h(y_0)\}$, it follows from Theorem 8(b) that $[Hf](y_0) = [H(f(h(y_0))\mathbf{1})](y_0)$. Since $y_0 \in Y_d$, there exists a sequence $f_n \in C(X, K)$ which converges in the norm to some $f \in C(X, K)$, but

$$Hf_n(y_0) = \left[H\left(f_n(h(y_0))\mathbf{1}\right)\right](y_0) \nrightarrow Hf(y_0) = \left[H\left(f(h(y_0))\mathbf{1}\right)\right](y_0).$$

Since $f_n(h(y_0))\mathbf{1} \to f(h(y_0))\mathbf{1}$, this contradicts the continuity of $H|_{[\mathbf{1}]}$ and proves (b). □

4. Bijections

We specialize to *bijective* additive separating maps H in this section. In many cases (Proposition 17) H being bijective implies that H is pointwise continuous (Definition 9). Since we know the form of $Hf(y)$ for $y \in Y_c$ (Proposition 11), this has a number of interesting consequences such as Theorems 18 and 19 below.

Proposition 17. *Let $H : C(X, K) \to C(Y, K)$ be an additive separating bijection. If X has an isolated point, suppose that $H|_{[\mathbf{1}]}$ is continuous. Then*:

(i) *If $K = \mathbb{Q}_p$ for some prime p or $K = \mathbb{R}$, then $H\mathbf{1}(y) \neq 0$ for every $y \in Y$.*

(ii) *If $K = \mathbb{C}$ or K is non-Archimedean and $K \neq \mathbb{Q}_p$ for any prime p, then $H(a\mathbf{1})(y) \neq 0$ for all nonzero $a \in K$ and $y \in Y$.*

Then H is pointwise continuous, i.e., $Y_c = Y$.

Proof. Let h be the support map of H. By Proposition 16, $h(Y_d) \subset h(Y_c)$. By Theorem 8(d) and the compactness of X, it follows that $h(Y) = h(Y_c \cup Y_d) = h(Y_c) = X$. Since Y_c is closed (Proposition 14), if there exists $y_0 \in Y_d$ then there exists $g \in C(Y, K)$ such that $g(Y_c) = \{0\}$ while $g(y_0) = 1$ by the complete regularity of Y for $K = \mathbb{R}$ or $\mathbb{C}$. If K is non-Archimedean, take g to be the K-valued characteristic function of a clopen neighborhood U of y_0 where U is disjoint from Y_c. Let $f \in C(X, K)$ be such that $Hf = g$. By Theorem 10,

$$Hf(y) = H\left[f\left(h(y)\right)\mathbf{1}\right](y) = g(y) = 0 \quad \text{for all } y \in Y_c.$$

If $K = \mathbb{R}$ or $\mathbb{Q}_p$, it follows by Proposition 11(a) that $f(h(y)) = 0$ for all $y \in Y_c$. Let $K = \mathbb{C}$ or any other non-Archimedean field. By (ii), $H(a\mathbf{1})$ never vanishes for $a \neq 0$ so Theorem 10 implies that $f(h(y)) = 0$ for all $y \in Y_c$. Thus, in any case, $f(x) = 0$ for all $x \in X$ ($f = 0$, in other words) and this contradicts $Hf(y_0) = g(y_0) = 1$. □

The Banach–Steinhaus theorem [15, Theorem 1, p. 84] for families of *linear* maps between normed linear spaces over a nontrivially valued field K asserts that

If $A_s : E \to F$ is a family of continuous linear maps that is pointwise bounded in the sense that for every point $x \in E$, $\sup_s \|A_s x\| < \infty$ then $\sup_s \|A_s\| < \infty$.

If $A : C(X, K) \to C(Y, K)$ is linear and K is nontrivially valued, it follows from the Banach–Steinhaus theorem that pointwise continuity of A ($y^\wedge \circ A$ continuous for all $y \in Y$) implies continuity of A. We reach a similar conclusion for separating additive bijections (as opposed to continuous linear maps).

Theorem 18. *$H : C(X, K) \to C(Y, K)$ denotes an additive separating pointwise continuous bijection and h is the support map of H. If $K \neq \mathbb{Q}_p$ is non-Archimedean $H|_{[\mathbf{1}]}$ is continuous.*

(i) *If $K = \mathbb{Q}_p$ for some prime p or $K = \mathbb{R}$, then $H\mathbf{1}(y) \neq 0$ for every $y \in Y$.*

(ii) *If $K = \mathbb{C}$ or $K \neq \mathbb{Q}_p$ for any prime p is non-Archimedean, suppose that $H(a\mathbf{1})(y) \neq 0$ for all nonzero $a \in K$ and $y \in Y$.*

Then H must be continuous and biseparating and its support map h is a homeomorphism of Y onto X. Moreover, in case K is

(a) *$\mathbb{R}$ or $\mathbb{Q}_p$, then $Hf(y) = f(h(y))H\mathbf{1}(y)$;*

(b) *$\mathbb{C}$, then $Hf(y) = (\mathrm{Re}[f(h(y))])H\mathbf{1}(y) + (\mathrm{Im}[f(h(y))])H(\mathrm{i}\mathbf{1})(y)$.*

Proof. Since H is onto, its support map h is 1–1 on $Y_c = Y$ by Proposition 12. Since X and Y are compact and $h(Y) = X$ (Theorem 8(d)), it follows that h is a homeomorphism.

If $K = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}_p$, then H is of the form of (a) or (b) above by Proposition 11 from which it follows that $H|_{[\mathbf{1}]}$ must be continuous. By these observations and (ii) it follows for all fields K that $H(a\mathbf{1})$ never vanishes if $a \neq 0$.

We now show that continuity of H follows from the continuity of $H|_{[\mathbf{1}]}$. By Proposition 13, there exists $k > 0$ such that $\|Ha\mathbf{1}\| \leqslant k\|a\mathbf{1}\|$. Thus if $f \in C(X, K)$,

$$\|Hf\| = \sup_{y\in Y} \bigl|Hf(y)\bigr| = \sup_{y\in Y} \bigl|H\bigl[f\bigl(h(y)\bigr)\mathbf{1}\bigr](y)\bigr| \leqslant \sup_{y\in Y} k\bigl\|f\bigl(h(y)\bigr)\mathbf{1}\bigr\| = k\|f\|$$

and H is continuous by Proposition 13.

Suppose $g_1, g_2 \in C(Y, K)$ are such that $g_1 g_2 = 0$. Choose $f_1, f_2 \in C(X, K)$ such that $Hf_1 = g_1$ and $Hf_2 = g_2$. Since $Hf(y) = H[f(h(y)\mathbf{1}](y)$, it follows that, for all $y \in Y$,

$$0 = g_1(y)g_2(y) = H\bigl[f_1\bigl(h(y)\bigr)\mathbf{1}\bigr](y)H\bigl[f_2\bigl(h(y)\bigr)\mathbf{1}\bigr](y).$$

Since $Ha\mathbf{1}$ never vanishes if $a \neq 0$, either $f_1(h(y)) = 0$ or $f_2(h(y)) = 0$. Since h is surjective, it follows that $f_1(x)f_2(x) = 0$ for all $x \in X$, i.e., $f_1 f_2 = 0$. Therefore H is biseparating. □

In Propositions 14, 16, and 17 we used the continuity of $H|_{[\mathbf{1}]}$ to show that $Y_c$is closed or $h(Y_d) \subset h(Y_c)$. And even though the continuity of $H|_{[\mathbf{1}]}$ was instrumental in proving that H is biseparating in Theorem 18,

Conjecture. *The continuity of $H|_{[\mathbf{1}]}$ is superfluous in Theorem* 18 *for non-Archimedean fields $K \neq \mathbb{Q}_p$.*

If H is an additive separating surjective isometry then $Y_c = Y$ and $H|_{[\mathbf{1}]}$ is continuous. It follows from Theorem 18 that the support map h is a homeomorphism and H is continuous and biseparating. Moreover, the form of $Hf(y)$ for $K = \mathbb{C}$ can be sharpened from that of Proposition 11(b) for surjective isometries. We deal with this in Theorem 19 below.

Theorem 19. *$H : C(X, K) \to C(Y, K)$ denotes an additive separating surjective isometry and h its support map. Furthermore, assume that*

(i) *If $K = \mathbb{Q}_p$ for some prime p or $K = \mathbb{R}$, then $H\mathbf{1}(y) \neq 0$ for every $y \in Y$.*

(ii) *If $K = \mathbb{C}$ or $K \neq \mathbb{Q}_p$ for any prime p is non-Archimedean, suppose that $H(a\mathbf{1})(y) \neq 0$ for all nonzero $a \in K$ and $y \in Y$.*

Then:

(a) *For any $y \in Y$, the map $a \mapsto [Ha\mathbf{1}](y)$ is an isometry of K.*

(b) *If $K = \mathbb{C}$ then there is a clopen set $U \subset Y$ such that $Hf(y) = f(h(y))H\mathbf{1}(y)$ for all $y \in U$ and $Hf(y) = \overline{f(h(y))}H\mathbf{1}(y)$ on $Y \backslash U$. Thus, if Y is connected then $Hf(y)$ is either $Hf(y) = f(h(y))H\mathbf{1}(y)$ or $Hf(y) = \overline{f(h(y))}H\mathbf{1}(y)$ for all $y \in U$.*

Proof. (a) Let a be a nonzero element of $K = \mathbb{R}$ or $\mathbb{C}$. Since H is an isometry, $|H(a\mathbf{1})(y)| \leqslant |a|$ for all $y \in Y$. Suppose $y_0 \in Y$ is such that $|Ha\mathbf{1}(y_0)| < |a|$. Choose $\varepsilon > 0$ such that $\varepsilon < (|a| - |H[a\mathbf{1}](y_0)|)/2$. Since $Ha\mathbf{1}$ is continuous, we may choose an open neighborhood W of y_0 such that $|H(a\mathbf{1})(y)| < |a| - \varepsilon$ for all $y \in W$. Since h is a homeomorphism, $V = h(W)$ is an open neighborhood of $h(y_0)$ and we may choose an open neighborhood U of $h(y_0)$ such that $\operatorname{cl} U \subset V$. Let $g \in C(X, K)$ be such that $0 \leqslant g(x) \leqslant 1$ for all $x \in X$, $g = 1$ on U, $g = 0$ on the complement $X \setminus V$. Therefore $H(ga\mathbf{1}) = H(a\mathbf{1})$ on $h^{-1}(U)$, $H(g\mathbf{1}) = 0$ on $Y \setminus [h^{-1}(V)]$ by Theorem 8(b). By the continuity of $y^\wedge \circ H$ for all $y \in Y$ (Proposition 17) and Proposition 11,

$$\big|H[ga\mathbf{1}](y)\big| = g\big(h(y)\big)\big|H(a\mathbf{1})(y)\big| \leqslant \big|H(a\mathbf{1})(y)\big|.$$

Because $|H(a\mathbf{1})(y)| < |a| - \varepsilon$ for $y \in h^{-1}(V)$ and $g(h(y)) = 0$ for $y \in h^{-1}(X \setminus V)$, then $\|H(ga\mathbf{1})\| < |a|$ while $\|ga\mathbf{1}\| = |a|$. This contradicts the fact that H is an isometry and the result is proved. For any non-Archimedean field K, we may take g to be the characteristic function of a clopen neighborhood W of $h(y_0)$ and use Theorem 10 instead of Proposition 11.

(b) By (a) it follows that $|H\mathbf{1}(y)| = |H(\mathrm{i}\mathbf{1})(y)| = 1$ and $|H[(1+\mathrm{i})\mathbf{1}](y)| = |1+\mathrm{i}| = \sqrt{2}$ for all $y \in Y$. Since $|H[(1+\mathrm{i})\mathbf{1}](y)| = |H\mathbf{1}(y) + H(\mathrm{i}\mathbf{1})(y)| = \sqrt{2}$, it follows that $H\mathbf{1}(y)$ and $H(\mathrm{i}\mathbf{1})(y)$ are orthogonal, i.e., $H(\mathrm{i}\mathbf{1})(y) = \pm \mathrm{i}H\mathbf{1}(y)$. Let $U = \{y \in Y\colon H(\mathrm{i}\mathbf{1})(y) = \mathrm{i}H\mathbf{1}(y)\}$ so that $H(\mathrm{i}\mathbf{1})(y) = -\mathrm{i}H\mathbf{1}(y)$ for $y \in Y \setminus U$. Since $H\mathbf{1}$ and $H(\mathrm{i}\mathbf{1})$ are continuous, U and $Y \setminus U$ are closed, therefore clopen. □

5. Examples

In Theorem 18 we assume when $K (\neq \mathbb{Q}_p)$ is non-Archimedean that the restriction $H|_{[\mathbf{1}]}$ of $H : C(X, K) \to C(Y, K)$ to the linear span $[\mathbf{1}]$ of $\mathbf{1}$ is continuous. This implies that H is continuous on a dense subalgebra of $C(X,K)$ (Proposition 21). Despite this, H need not be continuous on $C(X,K)$ (Example 22).

Lemma 20. *Let K be non-Archimedean. The linear span M of the collection of characteristic functions $\{k_U\colon U \subset X$ is clopen$\}$ is a dense subalgebra of $C(X,K)$.*

Proof. Let $f \in C(X,K)$ and $\varepsilon > 0$. For each $x \in X$ there exists a clopen neighborhood U_x of x such that if $z \in U_x$ then $|f(z) - f(x)| < \varepsilon$. From the ultrametric inequality it follows that $|f(z) - f(w)| < \varepsilon$ for all $z, w \in U_x$. As X is compact we may assume that there exist $x_i,\ i = 1, \ldots, n$, such that $X = \bigcup_{i=1}^n U_{x_i}$. Rewrite $\bigcup_{i=1}^n U_{x_i}$ as a union of pairwise disjoint sets $U_i,\ i = 1, 2, \ldots, n$. For each i, let $z_i \in U_i$. Since $|f(z) - f(w)| < \varepsilon$ for all $z, w \in U_i$ and $|f(z) - f(x_i)| < \varepsilon$ for all $z \in U_i$, it follows from the ultrametric inequality that

$$\left\| f - \sum_{i=1}^n f(z_i) k_{U_i} \right\| < \varepsilon.$$

Thus M is dense in $C(X,K)$. Since for $A, B \subset X$, $k_A k_B = k_{A \cap B}$, M is a subalgebra of $C(X,K)$. $\square$

Proposition 21. *Let K be non-Archimedean and let $H: C(X,K) \to C(Y,K)$ be an additive separating map. If $H|_{[\mathbf{1}]}$ is continuous, then H is continuous on the dense subalgebra of $C(X,K)$ generated by the characteristic functions of clopen subsets of X.*

Proof. Let k_U denote the characteristic function of the clopen subset $U \subset X$; let U' be the complement $X \setminus U$ of U. Clearly $\mathbf{1} = k_U + k_{U'}$ and k_U and $k_{U'}$ have disjoint cozero sets. Thus, if $a_n \to 0$ ($a_n \in K$), then $Ha_n\mathbf{1} = Ha_nk_U + Ha_nk_{U'} \to 0$. Since Ha_nk_U and $Ha_nk_{U'}$ have disjoint cozero sets, $Ha_nk_U \to 0$ and $Ha_nk_{U'} \to 0$. $\square$

Example 22. Let X be infinite. If K is non-Archimedean, then there exists a discontinuous additive separating map $H: C(X,K) \to C(Y,K)$ which is continuous on $[\mathbf{1}]$.

Proof. Let X be such that, for some $x_0 \in X$, not all functions in $C(X,K)$ are locally constant at x_0 [10, 4K2]. Let $f \in C(X,K)$ be not locally constant at x_0 with $f(x_0) = 0$. Let g be the separating discontinuous linear functional determined by f defined as follows (cf. [8, Example 3.6]). Let $\mathcal{F}$ be an ultrafilter of subsets of X containing the neighborhoods of x_0. The class M of functions $k \in C(X,K)$ such that $k = af$ for some $a \in K$ on some $A \in \mathcal{F}$ forms a linear subspace of $C(X,K)$. Let N be an algebraic complement of M in $C(X,K)$—i.e., a linear subspace N of $C(X,K)$ such that $M + N = C(X,K)$ and $M \cap N = \{0\}$ (cf. [12, p. 51]). We can choose N so that the linear span of characteristic functions of clopen subsets of X is contained in N. Write $w \in C(X,K)$ as $w = m + n$ for $m \in M$ and $n \in N$ where m and n are of course unique for a given w. Since $m \in M$, $m = af$ for some $a \in K$ on some $A \in \mathcal{F}$. We define a linear functional ϕ on $C(X,K)$ by taking $\phi(w) = \phi(m+n) = a$. For $y_0 \notin X$, topologize $Y = X \cup \{y_0\}$ by treating y_0 as an isolated point. For $w \in C(X,K)$, define $H'w(x) = w(x)$ for $x \in X$ and

$H'w(y_0) = \phi(w)$. $H' : C(X, K) \to C(Y, K)$ is a linear separating map; since $\phi(\mathbf{1}) = 0$, H' is continuous on $[\mathbf{1}]$.

We can construct an additive separating map H' that is not linear as follows. As shown in [7, Theorems 3.4–3.8], for all $K \neq \mathbb{Q}_p$, there exist non-linear, surjective, additive isometries $g : K \to K$. Define the additive functional λ at each $f \in C(X, K)$ as $\lambda(f) = g(\phi(f))$ where ϕ is as above. Then, with λ in place of ϕ in the definition of H', H' is separating [8, Example 3.6], discontinuous (ϕ is discontinuous and g is an isometry), additive and not linear. □

Example 23. If X has an isolated point and K is such that there is a discontinuous additive 1–1 map g from K onto itself, then
(a) there exists a discontinuous biseparating bijection $H : C(X, K) \to C(Y, K)$,
(b) the support map h of H is a homeomorphism,
(c) there exists $y \in Y$ such that $y^\wedge \circ H$ is of the form of Theorem 10 but is discontinuous.

Proof. Let x_0 be an isolated point of the compact Hausdorff space X. With $Y = X$, for $f \in C(X, K)$, let $Hf(y) = f(y)$ for $y \neq x_0$ while $Hf(x_0) = g(f(x_0))$. Clearly H is separating, and additive. As x_0 is isolated, the value of $f(x_0)$ is independent of the values that f assumes on $X - \{x_0\}$. Therefore H is bijective and its support map h is readily seen to be the identity map. By its definition, H is biseparating. Now consider the form of $x_0{}^\wedge \circ H$. Consider $f \in C(X, K)$ so $Hf(x_0) = g(f(x_0)) = H(f(x_0)\mathbf{1})(x_0)$. Since g is discontinuous, $x_0{}^\wedge \circ H$ is discontinuous. Clearly $H|_{[\mathbf{1}]} = g$ is discontinuous. □

Example 24. Some additive biseparating nonlinear isometries of $C(X, K)$ onto $C(Y, K)$ that are not weighted composition maps.

Proof. For all complete nontrivially valued K other than $\mathbb{Q}_p$ there exist nonlinear additive surjective isometries $k : K \to K$ [7, Theorems 3.4–3.8]. Let X and Y be homeomorphic (via a homeomorphism $g : Y \to X$) compact 0-dimensional Hausdorff spaces. We claim that $H_k : C(X, K) \to C(Y, K)$, $f \mapsto k \circ f \circ g$, is an additive nonlinear isometry that is not a weighted composition map. Clearly $Hf \in C(Y, K)$ for all $f \in C(X, K)$. Since k is an additive isometry, so is H_k. Since, by the way k is constructed, $H_k(a\mathbf{1})(y) = k(a) \neq ak(1)$, for some $a \in K$ it follows that $k(a)\mathbf{1} = H_k(a\mathbf{1}) \neq aH_k\mathbf{1} = ak(1)\mathbf{1}$ so H_k is not a weighted composition. To see that H_k is onto, let $w \in C(Y, K)$. Since k is a surjective isometry, $k^{-1} \circ w \circ g^{-1} \in C(X, K)$ and $w = H_k(k^{-1} \circ w \circ g^{-1})$.

We can construct other such maps as follows. Let $U_1, \ldots, U_n$ be a clopen partition of X and let $g : Y \to X$ be a surjective homeomorphism. Let $k_1, \ldots, k_n$ be distinct additive surjective isometries of K (there are infinitely many by [7, Theorems 3.4–3.8] which are not unit multipliers. Now define for $f \in C(X, K)$, $H_{k_1,\ldots,k_n}(f) = k_i \circ f \circ g$ on $g^{-1}(U_i)$ for $i = 1, \ldots, n$. Since the k_i are not linear, $H_{k_1,\ldots,k_n}(f)$ is not a weighted composition map. □

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